Symplectic Quasi-States and Semi-Simplicity of Quantum Homology

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Abstract

We review and streamline our previous results and the results of Y. Ostrover on the existence of Calabi quasi-morphisms and symplectic quasi-states on symplectic manifolds with semi-simple quantum homology. As an illustration, we discuss the case of symplectic toric Fano 4-manifolds. We present also new results due to D. McDuff: she observed that for the existence of quasi-morphisms/quasi-states it suffices to assume that the quantum homology contains a field as a direct summand, and she showed that this weaker condition holds true for one point blow-ups of non-uniruled symplectic manifolds.

1 Symplectic quasi-states and Calabi quasi-morphisms

SYMPLECTIC QUASI-STATES: Let (M^{2n}, ω) be a closed connected $2n$-dimensional symplectic manifold. A *symplectic quasi-state* on M is a (possibly non-linear!) real-valued functional ζ on the space $C(M)$ of all continuous functions on M which satisfies the following conditions:

Quasi-linearity: $\zeta(F + \lambda G) = \zeta(F) + \lambda \zeta(G)$ for all $\lambda \in \mathbb{R}$ and for all functions $F, G \in C^\infty(M)$ which commute with respect to the Poisson brackets: $\{F, G\} = 0$.

Monotonicity: $\zeta(F) \leq \zeta(G)$ if $F \leq G$.

Normalization: $\zeta(1) = 1$.

Non-linear quasi-states are known to exist on certain symplectic manifolds. For instance, we will prove in this paper that they exist on any symplectic toric Fano four-manifold equipped with a rational symplectic form. Here, denoting by $H_2^S(M)$ the group of spherical integral homology classes of M, we will say that ω is *rational* if the subgroup $[\omega](H_2^S(M)) \subset \mathbb{R}$ is discrete. The list of symplectic toric Fano four-manifolds consists of $\mathbb{S}^2 \times \mathbb{S}^2$ and the blow-ups of $\mathbb{C}P^2$ at k, $0 \le k \le 3$, points equipped with the usual structures of a symplectic toric manifold (see Theorem 1.1 and Section 4 below). Another class of examples of symplectic manifolds admitting a non-linear symplectic quasi-state was discovered recently by D. McDuff [28]: it includes a one point blow-up of any non-uniruled closed symplectic manifold (for instance, of the standard symplectic four-torus $\mathbb{T}^4$), see Section 3 below for more details. However we do not know at the moment whether a non-linear symplectic quasi-state exists on $\mathbb{T}^4$ (without blowing up)! In fact, there is only one currently known way of constructing non-linear symplectic quasi-states on symplectic manifolds of dimension higher than 2. It is based on the Floer theory for periodic orbits of Hamiltonian flows and will be discussed in more detail below. Beyond it the question of existence of a non-linear symplectic quasi-state on a given symplectic manifold of dimension greater than two is completely open.

Let us mention that on two-dimensional manifolds symplectic quasi-states exist in abundance. They are supplied by the theory of topological quasi-states, a beautiful recently emerged branch of functional analysis whose foundations were laid by Aarnes in [1].

Non-linear symplectic quasi-states are interesting for several reasons.

The first reason is related to a mathematical model of quantum mechanics which goes back to von Neumann [46]. The basic objects in this model are *observables* and *states*. Observables are Hermitian operators on a complex Hilbert space V. They form a real Lie algebra $\mathcal{A}_q$ (q for quantum) with the Lie bracket given by $[A, B]_\hbar = \frac{i}{\hbar}(AB - BA)$, where $\hbar$ is the Planck constant. States are real-valued linear functionals on $\mathcal{A}_q$ that are positive (i.e take non-negative values on non-negative Hermitian operators) and normalized (that is equal 1 on the identity operator). Observables represent physical quantities associated to a physical system such as energy, position, momentum etc. while the value of a state on an observable represents the mathematical expectation for the value of such a physical quantity in the given state of

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the physical system.

A number of prominent physicists disagreed with the linearity axiom put by von Neumann in the definition of a state. Their reasoning was that the additivity of a state on two observables A,B makes sense *a priori* only if the corresponding physical quantities are simultaneously measurable, meaning that A and B commute: $[A, B]_\hbar = 0$. However in 1957 Gleason [19] proved a remarkable theorem implying, under the assumption $\dim V \geq 3$, that even if one considers a positive functional on $\mathcal{A}_q$ which is only required to be linear on any pair of commuting observables then it has to be globally linear. In other words, each "quantum quasi-state" has to be a state.

In the analogous mathematical model of classical mechanics the algebra $\mathcal{A}_c$ of observables (c for classical) is the space of continuous functions $C(M)$ on a symplectic manifold (M, ω). The Lie bracket is defined as the Poisson bracket on the dense subspace $C^\infty(M) \subset C(M)$. States are linear functionals on $\mathcal{A}_c$ that are positive (i.e. take non-negative values on non-negative functions) and normalized (equal 1 on the constant function 1). Thus the existence of non-linear symplectic quasi-states exhibits what can be called an "anti-Gleason" phenomenon in classical mechanics: not every "classical quasi-state" is a state [13], [14].

The second point of interest in symplectic quasi-states is related to the C^0-rigidity of the Poisson brackets discovered by Cardin and Viterbo [7]: while the definition of the Poisson brackets $\{F, G\}$ involves the first derivatives of F and G, it turns out that when the function $\{F, G\}$ is not identically zero its C^0-norm cannot be made arbitrarily small by arbitrarily C^0-small perturbations of F and G. Symplectic quasi-states can be used to give effective quantitative expressions for this and similar rigidity phenomena – see [14] for details.

Thirdly, applications of symplectic quasi-states include results on rigidity of intersections – a well-known phenomenon in symplectic topology, meaning that a subset of a symplectic manifold (M, ω) cannot be completely displaced from another subset by a symplectic isotopy while it is possible to do it by a smooth one. In fact, in all known applications of this sort it suffices to have a functional on $C(M)$ with weaker properties, called a *partial symplectic quasi-state*, whose definition won't be given here – see [13], [15] for details. However in case the symplectic quasi-state ζ comes from the Floer theory construction the reason it yields the rigidity of intersections becomes really basic and simple [13]: by a generalized Riesz representation theorem of Aarnes [1] to each symplectic quasi-state ζ one can associate a measure-type set-function

τ_ζ, called a *quasi-measure* (or a *topological measure*), which is finitely additive (but not finitely subadditive unless ζ is linear). Moreover, such a symplectic quasi-state ζ is invariant with respect to the natural action of the group[1] $Symp_0(M)$ on $C(M)$ ("Invariance property") and therefore τ_ζ is invariant under symplectic isotopies. Hence if the sum of the quasi-measures of two subsets of M is greater than the quasi-measure of the whole manifold M none of these subsets can be displaced from the other by a symplectic isotopy. In order to compute the quasi-measures one may use in certain cases the following important property of ζ: it vanishes on functions with sufficiently small support. More precisely, we call a set $U \subset M$ *displaceable* if there exists a Hamiltonian symplectomorphism[2] ψ such that $\psi(U) \cap \text{Closure}\, U = \emptyset$. Then $\zeta(F) = 0$ for any $F \in C(M)$ such that $\text{supp}\, F$ is displaceable ("Vanishing property").

CALABI QUASI-MORPHISMS: Fourthly, any symplectic quasi-state ζ coming from the Floer theory construction mentioned above is an "infinitesimal version" of a function $\mu : \widetilde{Ham}(M) \to \mathbb{R}$ on the universal cover $\widetilde{Ham}(M)$ of the group $Ham(M)$ of Hamiltonian symplectomorphisms of M whose Lie algebra can be identified with $C^\infty(M)/\mathbb{R}$. We shall need the following notation: Given an (in general, time-dependent) Hamiltonian $F = F_t$, $0 \le t \le 1$, on M denote by ϕ_F an element of $\widetilde{Ham}(M)$ represented by the time-1 Hamiltonian flow generated by F and viewed as an identity-based path in $Ham(M)$. With this notation, for every $F \in C^\infty(M)$

$$\zeta(F) = \frac{\int_M F\omega^n}{\text{vol}\,(M)} - \frac{\mu(\phi_F)}{\text{vol}\,(M)}, \tag{1}$$

where $\text{vol}\,(M) = \int_M \omega^n$ is the symplectic volume of M.

The function μ is of interest on its own merit – it is a *homogeneous quasi-morphism*, i.e. it satisfies the following properties:

Quasi-additivity: There exists $C > 0$, which depends only on μ, so that

$$|\mu(\phi\psi) - \mu(\phi) - \mu(\psi)| \le C \quad \text{for } \textit{all} \text{ elements } \phi, \psi \in \widetilde{Ham}(M).$$

Homogeneity: $\mu(\phi^m) = m\mu(\phi)$ for each $\phi \in \widetilde{Ham}(M)$ and each $m \in \mathbb{Z}$.

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[1]The group $Symp_0(M)$ is the identity component of the group $Symp\,(M)$ of symplectomorphisms of M.

[2]See e.g. [29], [36] for preliminaries on Hamiltonian symplectomorphisms.

Homogeneous quasi-morphisms on groups have become increasingly important objects in various fields of mathematics (see e.g. [18]), especially in the case when a group does not admit a non-trivial real-valued homomorphism and thus homogeneous quasi-morphisms are, in a sense, the closest approximation to a homomorphism one can hope to get on such a group. This is precisely the case for $\widetilde{Ham}(M)$ which is a perfect group by Banyaga's theorem [2] and therefore does not admit a non-trivial real-valued homomorphism. Moreover, the homogeneous quasi-morphism μ, constructed by means of the Floer theory, has an additional remarkable property: it "patches up" a family of homomorphisms on a certain class of subgroups of $\widetilde{Ham}(M)$ in the following sense. For any open $U \subset M$, $U \neq M$, define a subgroup $\widetilde{Ham}(U)$ of $\widetilde{Ham}(M)$ as

$$\widetilde{Ham}(U) := \{\, \phi_F \in \widetilde{Ham}(M) \mid \mathrm{supp}(F_t) \subset U \text{ for all } t\}.$$

Assuming that F_t is supported in U for every t, the formula

$$\phi_F \mapsto \int_0^1 dt \int_M F_t \omega^n$$

yields a correctly defined function $Cal_U : \widetilde{Ham}(U) \to \mathbb{R}$ which is moreover a homomorphism, called the *Calabi homomorphism* (see [2, 29]). Then μ satisfies the following property corresponding on the infinitesimal level to the vanishing property of ζ:

Calabi property: $\mu|_{\widetilde{Ham}(U)} = Cal_U$ for any displaceable open $U \subset M$.

A homogeneous quasi-morphism on $\widetilde{Ham}(M)$ satisfying the Calabi property is called a *Calabi quasi-morphism*. In particular, we can now state

Theorem 1.1. *Assume M is a symplectic toric Fano 4-manifold with a rational symplectic structure. Then M admits a Calabi quasi-morphism μ and a symplectic quasi-state ζ which satisfies the vanishing and the invariance properties, with μ and ζ being related to each other by (1).*

For applications of the Calabi quasi-morphisms coming from the Floer theory to the study of the Hofer geometry on $Ham(M)$ and the algebraic structure of $\widetilde{Ham}(M)$ and $Ham(M)$ see [12]. Let us mention also that Py constructed Calabi quasi-morphisms for closed surfaces of genus ≥ 1 using completely different ideas, see [37, 38].

Let us mention that in [13] we first introduced the Calabi quasi-morphism μ, then defined the symplectic quasi-state ζ on the space $C^\infty(M)$ of smooth functions by formula (1), and afterwards extended it to the space $C^{(}M)$ of continuous functions by the continuity in the uniform norm. We give a concise definition of ζ in formula (7) below.

2 Quantum homology

Abusing the notation, we will write $\omega(A)$, $c_1(A)$ for the results of evaluation the cohomology classes $[\omega]$ and $c_1(M)$ on $A \in H_2(M; \mathbb{Z})$. Set

$$\bar{\pi}_2(M) := H_2^S(M)/\sim,$$

where by definition

$$A \sim B \text{ iff } \omega(A) = \omega(B) \text{ and } c_1(A) = c_1(B).$$

Denote by $\Gamma(M, \omega) := [\omega](H_2^S(M)) \subset \mathbb{R}$ the subgroup of periods of the symplectic form on M on spherical homology classes.

Novikov ring: Let $\Gamma \subset \mathbb{R}$ be a countable subgroup (with respect to the addition). Let s, q be formal variables. Let $\mathcal{F}$ be a basic field of coefficients (in applications, one usually takes $\mathcal{F} = \mathbb{C}$ or $\mathcal{F} = \mathbb{Z}_2$). Define a field $\mathcal{K}_\Gamma$ whose elements are generalized Laurent series in s of the following form:

$$\mathcal{K}_\Gamma := \left\{ \sum_{\theta \in \Gamma} z_\theta s^\theta, \ z_\theta \in \mathcal{F}, \ \sharp\{\theta > c \mid z_\theta \neq 0\} < \infty, \ \forall c \in \mathbb{R} \right\}.$$

Define a ring $\Lambda_\Gamma := \mathcal{K}_\Gamma[q, q^{-1}]$ as the ring of polynomials in q, q^{-1} with coefficients in $\mathcal{K}_\Gamma$. We turn Λ_Γ into a graded ring by setting the degree of s to be zero and the degree of q to be 2. Note that the grading on Λ_Γ takes only even values. The ring Λ_Γ serves as an abstract model of the Novikov ring associated to a symplectic manifold.

By definition, the Novikov ring of a symplectic manifold (M, ω) is $\Lambda_{\Gamma(M,\omega)}$. In what follows, when (M, ω) is fixed, we abbreviate and write Γ, $\mathcal{K}$ and Λ instead of $\Gamma(M, \omega)$, $\mathcal{K}_{\Gamma(M,\omega)}$ and $\Lambda_{\Gamma(M,\omega)}$ respectively.

Quantum homology ring - definition: The quantum homology ring $QH_*(M)$ is defined as follows. First, it is a graded module over Λ given by

$$QH_*(M) := H_*(M; \mathcal{F}) \otimes_{\mathcal{F}} \Lambda,$$

with the grading defined by the gradings on $H_*(M; \mathcal{F})$ and Λ:

$$\deg(a \otimes z s^\theta q^k) := \deg(a) + 2k.$$

Second, and most important, the Λ-module $QH_*(M)$ is equipped with a *quantum product*: given $a \in H_k(M; \mathcal{F})$, $b \in H_l(M; \mathcal{F})$, their quantum product is a class $a * b \in QH_{k+l-2n}(M)^3$, defined by

$$a * b = \sum_{A \in \bar{\pi}_2(M)} (a * b)_A \otimes s^{-\omega(A)} q^{-c_1(A)},$$

where $(a * b)_A \in H_{k+l-2n+2c_1(A)}(M; \mathcal{F})$ is defined by the requirement

$$(a * b)_A \circ c = GW_A^{\mathcal{F}}(a, b, c) \ \forall c \in H_*(M; \mathcal{F}).$$

Here $\circ$ stands for the intersection index and $GW_A^{\mathcal{F}}(a, b, c) \in \mathcal{F}$ denotes the Gromov-Witten invariant which, roughly speaking counts the number of pseudo-holomorphic spheres in M in the class A that meet cycles representing $a, b, c \in H_*(M; \mathcal{F})$ (see [39], [40], [30] for the precise definition).

Extending this definition by Λ-linearity to the whole $QH_*(M)$ one gets a correctly defined graded-commutative associative product operation $*$ on $QH_*(M)$ which is a deformation of the classical $\cap$-product in singular homology [25], [30], [39], [40], [47]. The *quantum homology algebra* $QH_*(M)$ is a ring whose unity is the fundamental class $[M]$ and which is a module of finite rank over Λ. If $a, b \in QH_*(M)$ have graded degrees $deg(a)$, $deg(b)$ then

$$deg(a * b) = deg(a) + deg(b) - 2n.$$

Denote by $H_{ev}(M; \mathcal{F})$ the even-degree part of the singular homology and by $QH_{ev}(M) = H_{ev}(M; \mathcal{F}) \otimes_{\mathcal{F}} \Lambda$ the even-degree part of $QH_*(M)$. Then $QH_{ev}(M)$ is a commutative subring of $QH_*(M)$ which is a module of finite rank over Λ. We will identify Λ with a subring of $QH_{ev}(M)$ by $\lambda \mapsto [M] \otimes \lambda$.

SEMI-SIMPLE ALGEBRAS: Recall that a commutative (finite-dimensional) algebra Q over a field $\mathcal{B}$ is called *semi-simple* if it splits into a direct sum of fields as follows: $Q = Q_1 \oplus ... \oplus Q_d$, where

- each $Q_i \subset Q$ is a finite-dimensional linear subspace over $\mathcal{B}$;

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[3]Recall that $2n = \dim M$.

- each Q_i is a field with respect to the induced ring structure;

- the multiplication in Q respects the splitting:

$$(a_1, ..., a_d) \cdot (b_1, ..., b_d) = (a_1 b_1, ..., a_d b_d).$$

A classical theorem of Wedderburn (see e.g. [45]) implies that the semi-simplicity is equivalent to the absence of nilpotents in the algebra.

Let Q be a commutative algebra over a field $\mathcal{B}$. Assume the base field $\mathcal{B}$ is a subfield of a larger field $\bar{\mathcal{B}}$. Then Q extends to a $\bar{\mathcal{B}}$-algebra $\bar{Q}$:

$$\bar{Q} := Q \otimes_{\mathcal{B}} \bar{\mathcal{B}}.$$

One can identify Q with a $\mathcal{B}$-subalgebra of $\bar{Q}$ (whenever the latter is viewed as a $\mathcal{B}$-algebra).

Proposition 2.1.

(A) Assume the field $\mathcal{B}$ is of characteristic zero. Then the $\mathcal{B}$-algebra Q is semi-simple if and only if the $\bar{\mathcal{B}}$-algebra $\bar{Q}$ is semi-simple.

(B) A tensor product of finite-dimensional semi-simple algebras over a field of characteristic 0 is again semi-simple.

For the proof see e.g. [45], §103. The "if" part of (A) is trivial and does not require any assumptions on the characteristic of $\mathcal{B}$: if there were a nilpotent in Q then it would have been a nilpotent also in $\bar{Q}$.

A remark on part (A) of the proposition: Note that the fields in the decomposition of the algebra, even their number, may change and the unity of such a field may no longer be one after passing from Q to $\bar{Q}$.

Semi-simplicity is a partial case of the following more general algebraic property which is relevant to our discussion – see Section 3 below. Namely, we say a commutative finite-dimensional algebra Q over a field $\mathcal{B}$ *contains a field as a direct summand* if it decomposes as a $\mathcal{B}$-algebra into *a direct sum of $\mathcal{B}$-algebras*

$$Q = Q_1 \oplus Q_2 , \quad \text{where} \quad Q_1 \quad \text{is a field} . \tag{2}$$

Let us emphasize that no assumption on Q_2 is imposed.

The next proposition generalizes Proposition 2.1. Though it is not used in the sequel of the paper, it highlights an important feature of the property of containing a field as a direct summand: this property turns out to be robust

with respect to extensions of the base fields. This feature is highly desirable in the context of quantum homology where one faces the necessity to choose the base field depending upon the circumstances. As before, assume $\mathcal{B}$ is a subfield of a larger field $\bar{\mathcal{B}}$ and extend Q to a $\bar{\mathcal{B}}$-algebra $\bar{Q}$: $\bar{Q} := Q \otimes_{\mathcal{B}} \bar{\mathcal{B}}$.

Proposition 2.2.

(A) Assume the field $\mathcal{B}$ is of characteristic zero. Then the $\mathcal{B}$-algebra Q contains a field as a direct summand if and only if the $\bar{\mathcal{B}}$-algebra $\bar{Q}$ does.

(B) Assume two finite-dimensional commutative algebras over a field of characteristic 0 both contain fields as direct summands. Then their tensor product also has this property.

The "only if" part of (A) and part (B) follow readily from Proposition 2.1. The "if" part of (A) is less trivial and will be proved now for the sake of completeness.

Proof.

PRELIMINARIES: First of all, it suffices to prove the theorem for algebraically closed $\bar{\mathcal{B}}$: indeed, by part (A) of Proposition 2.1, if $Q \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ contains a field as a direct summand, the same holds for $Q \otimes_{\mathcal{B}} \bar{\mathcal{B}}'$ where $\bar{\mathcal{B}}'$ is the algebraic closure of $\bar{\mathcal{B}}$. In what follows we assume that $\bar{\mathcal{B}}$ is algebraically closed.

We shall use notation $A \oplus B$ for the direct sum of *algebras* (over the same base field) and $A \dotplus B$ for the direct sum of *modules* (also over the same field).

Given extensions $\mathcal{B} \subset E \subset \bar{\mathcal{B}}$ of fields, we shall say that an element $t \in Q \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ is *defined over* E if it can be written as $t = \sum q_j \otimes \theta_j$ with $q_j \in Q, \theta_j \in E$.

For $q \in Q$ we write $\hat{q} := q \otimes 1_{\bar{\mathcal{B}}}$, where $1_{\bar{\mathcal{B}}}$ is the unity of $\bar{\mathcal{B}}$.

Every semi-simple commutative finite-dimensional algebra over an algebraically closed field admits a unique decomposition into one-dimensional fields. The correspondent unities (called *basic idempotents*) sum up to the unity of the algebra (we refer to this as to *the partition of unity*). By the Krull-Schmidt theorem, the basic idempotents are unique up to permutation.

STARTING THE PROOF: By the Wedderburn principal theorem (see e.g. [17], Thm 13.18)

$$Q = (F_1 \oplus ... \oplus F_n) \dotplus Y ,$$

where Y is the radical [4] and all F_i are fields. Consider any monomorphism of F_i into $\bar{\mathcal{B}}$ and denote by $E_i \subset \bar{\mathcal{B}}$ the normal closure of the image. The field E_i does not depend on the choice of a monomorphism. A concrete realization of E_i is as follows. Take any primitive element $a_i \in F_i$ and denote by P_i its minimal polynomial. Consider the factorization of P_i in $\bar{\mathcal{B}}$ into linear factors:

$$P_i(u) = (u - \mu_{i1}) \cdot ... \cdot (u - \mu_{ir_i}) \ .$$

All μ_j's are pair-wise distinct (since F_i is a field). Then

$$E_i = \mathcal{B}(\mu_{i1}, ..., \mu_{ir_i}) \ .$$

Lemma 2.3. *The basic idempotents of $F_i \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ are defined over E_i for all $i = 1, ..., n$.*

PROOF OF THE LEMMA. We use the notation introduced above, and omit the lower index i. Given $j \in \{1; ...; r\}$, introduce the polynomial $t_j(u)$ of degree $r - 1$ which vanishes at all μ_m for $m \neq j$ and equals 1 at μ_j. One immediately checks that its coefficients lie in E. Note that the basic idempotents of $F \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ are given by $t_j(a)$: In order to see this use that $F = \mathcal{B}[a] = \mathcal{B}[u]/(P)$ and hence $F \otimes_{\mathcal{B}} \bar{\mathcal{B}} = \bar{\mathcal{B}}[u]/(P)$. The lemma follows. $\square$

Denote by G_i the Galois group of E_i over $\mathcal{B}$. We extend its action to $Q \otimes_{\mathcal{B}} E_i$ by the change of coefficients:

$$g(q \otimes \theta) = q \otimes g(\theta) \ \ \forall q \in Q, \ \theta \in E_i, \ g \in G_i \ .$$

Lemma 2.4. *The Galois group G_i acts transitively on basic idempotents of $F_i \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ for all $i = 1, ..., n$.*

PROOF OF THE LEMMA. Again, we omit the lower index i. Take any basic idempotent, say e_1 and denote by $\mathcal{O}$ its orbit under the action of G. Then

$$h := \sum_{f \in \mathcal{O}} f \tag{3}$$

is an idempotent *invariant under the action of G*. Indeed, the invariance is obvious and in order to prove that h is an idempotent, note that, according

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[4]For finite-dimensional commutative algebras the radical is the set of all nilpotent elements.

to Lemma 2.3, each $g \in G$ preserves the set of basic idempotents of $F \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ and therefore h is the sum of the basic idempotents lying in the orbit of the action of G on that set, which implies that h itself is an idempotent.

Moreover, h is defined over $\mathcal{B}$. Therefore it lies in $F \otimes_{\mathcal{B}} \mathcal{B}$. The latter is isomorphic to F, which was assumed to be a field. Thus $h = \widehat{1_F}$. We conclude that formula (3) gives the partition of unity in $F \otimes_{\mathcal{B}} \bar{\mathcal{B}}$. Thus the sum at the right hand side of the formula includes *all* the basic idempotents, and the lemma follows. $\qquad\square$

EXPLORING THE DIRECT SUMMAND: Let $H \subset Q \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ be the field (the direct summand) from the assumption of the theorem. Denote by e its unity.

Lemma 2.5. *Element e coincides with one of the basic idempotents of some* $F_i \otimes_{\mathcal{B}} \bar{\mathcal{B}}$.

PROOF OF THE LEMMA. Since

$$Q \otimes_{\mathcal{B}} \bar{\mathcal{B}} = (F_1 \otimes_{\mathcal{B}} \bar{\mathcal{B}} \oplus ... \oplus F_n \otimes_{\mathcal{B}} \bar{\mathcal{B}}) \dotplus Y \otimes_{\mathcal{B}} \bar{\mathcal{B}}$$

we can decompose

$$e = (x_1 \oplus ... \oplus x_n) \dotplus y$$

with $x_i \in F_i \otimes_{\mathcal{B}} \bar{\mathcal{B}}, y \in Y \otimes_{\mathcal{B}} \bar{\mathcal{B}}$.

Since $e^2 = e$ we have

$$(x_1^2 \oplus ... \oplus x_n^2) \dotplus (2x_1 y + ... + 2x_n y + y^2) = (x_1 \oplus ... \oplus x_n) \dotplus y \ .$$

Thus

$$x_i^2 = x_i \ \ \forall i = 1, ..., n \tag{4}$$

and

$$2x_1 y + ... + 2x_n y + y^2 = y \ . \tag{5}$$

Multiplying (5) by x_i we have $x_i y + x_i y^2 = 0$. Rewrite this as $x_i y + (x_i y)^2 = 0$. Since $x_i y$ is a nilpotent, this yields $x_i y = 0$. Substituting this into (5) we get $y^2 = y$, and since y is nilpotent we have $y = 0$.

Let us write x_i as a linear combination of the basic idempotents e_{ij}, $j = 1, ..., r_i$, of $F_i \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ and substitute into (4). We get that

$$e = \sum_{i=1}^{n} \sum_{j \in S_i} e_{ij} \tag{6}$$

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for some subsets $S_i \subset \{1; ...; r_i\}$. We claim that the sum in the right hand side of (6) consists of one term only: Indeed, assume on the contrary that there are at least two distinct terms, say e_{ab} and e_{cd}. Then $e_{ab} = e_{ab}e$ and $e_{cd} = e_{cd}e$ and therefore both e_{ab} and e_{cd} lie in H (here we use that H is a direct summand in the sense of algebras). But $e_{ab} \cdot e_{cd} = 0$, and we get a contradiction with the fact that the field H has no divisors of zero. The claim, and thus the lemma, follow. $\square$

THE END OF THE PROOF OF THE PROPOSITION: In view of Lemma 2.5, we can assume without loss of generality that the unity element e of H equals e_{11}.

Since H is a direct summand and a field, $\widehat{y} \cdot e_{11} = 0$ for every y in the radical Y. We claim that $\widehat{y} \cdot e_{1j} = 0$ for all $j = 1, ..., r_1$. Indeed, for given j by Lemma 2.4 there is an element, say, g of the Galois group G_1 with $g(e_1) = e_j$. Note that G_1 fixes $\widehat{q}$ for every $q \in Q$. Thus

$$\widehat{y} \cdot e_{1j} = g(\widehat{y} \cdot e_{11}) = 0 \ ,$$

and the claim follows. Since

$$e_{11} + ... + e_{1r_1} = \widehat{1_{F_1}}$$

we have that

$$\widehat{y \cdot 1_{F_1}} = \widehat{y} \cdot \widehat{1_{F_1}} = 0 \ \ \forall y \in Y \ .$$

Thus $Y \cdot F_1 = 0$ in Q, which yields the decomposition

$$Q = F_1 \oplus (F_2 \oplus ... \oplus F_n \dotplus Y) \ .$$

We conclude that F_1 is the desired field summand of Q. This completes the proof. $\square$

3 From quantum homology to quasi-states

SPECTRAL INVARIANTS: Let us now discuss in more detail the construction of ζ via Hamiltonian Floer theory. The construction involves *spectral numbers* $c(a, H)$ and $c(a, \phi)$ associated by means of certain minimax-type procedure in the Floer theory to any non-zero quantum homology class a, any (time-dependent) Hamiltonian H on M [34] and any element $\phi \in \widetilde{Ham}(M, \omega)$ (see

also [44, 31, 41, 32] for earlier versions of this theory). We refer to [15] for a detailed review.

In what follows we assume that all the symplectic manifolds we deal with belong to the class $\mathcal{S}$ of closed symplectic manifolds for which the spectral invariants are well-defined and enjoy the standard list of properties (see e.g. [30, Theorem 12.4.4]). For instance, $\mathcal{S}$ contains all spherically monotone manifolds, that is symplectic manifolds (M, ω) such that $[\omega]|_{H_2^S(M)} = \kappa \, c_1|_{H_2^S(M)}$ for some $\kappa > 0$, as well as manifolds with $[\omega]|_{H_2^S(M)} = 0$. Furthermore, $\mathcal{S}$ contains all symplectic manifolds M^{2n} for which, on one hand, either $c_1 = 0$ or the minimal Chern number (i.e. the positive generator of $c_1(H_2^S(M)) \subset \mathbb{Z}$) is at least $n-1$ and, on the other hand, $[\omega]$ is rational. This includes all symplectic 4-manifolds with a rational symplectic structure. The general belief is that the class $\mathcal{S}$ includes **all** symplectic manifolds[5].

Denote by $QH_{2n}(M)$ the graded component of degree $2n$ of the quantum homology algebra $QH_*(M)$. It is an algebra over the field $\mathcal{K}$. We will say that $QH_{2n}(M)$ is semi-simple, if it is semi-simple as a $\mathcal{K}$-algebra. Elementary grading considerations show that any non-zero idempotent in $QH_*(M)$ has to lie in $QH_{2n}(M)$. Given an non-zero idempotent $a \in QH_{2n}(M)$, a time-independent Hamiltonian $H : M \to \mathbb{R}$ and an element $\phi \in \widetilde{Ham}(M)$, define

$$\zeta(H) := \lim_{l \to +\infty} \frac{c(a, lH)}{l}, \tag{7}$$

and

$$\mu(\phi) := -\text{vol}\,(M) \cdot \lim_{l \to +\infty} \frac{c(a, \phi^l)}{l}, \tag{8}$$

where $\text{vol}\,(M) := \int_M \omega^n$. For manifolds from class $\mathcal{S}$ formula (7) yields a functional on $C^\infty(M)$ which can be extended to $C(M)$. The link between existence of symplectic quasi-states and Calabi quasi-morphisms on the one hand, and semi-simplicity of quantum homology on the other hand, is given by the next theorem.

Theorem 3.1. *Assume that $QH_{2n}(M)$ is semi-simple and decomposes into a direct sum of fields: $QH_{2n}(M) = Q_1 \oplus \ldots \oplus Q_d$. Assume the idempotent a appearing in the definitions of ζ and μ (see equations (7), (8)) is the unit*

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[5]According to the recent preprint of M.Usher [43] and the previous result of Y.-G.Oh [33] on the "non-degenerate spectrality" property of spectral numbers, the rationality assumption on $[\omega]$ is not needed for the further results in this paper.

element in some Q_i. Then μ is a Calabi quasi-morphism and ζ is a symplectic quasi-state satisfying the vanishing and the invariance properties.

THE ROLE OF SEMI-SIMPLICITY: The example $M = \mathbb{T}^2$ [13] shows that the semi-simplicity assumption cannot be completely omitted – without it ζ may turn out to be only a partial symplectic quasi-state (a weaker object already mentioned above) and not a genuine one. On the other hand, as it was pointed out to us by D. McDuff [28], it suffices to assume that the $\mathcal{K}$-algebra $QH_{2n}(M)$ contains a field as a direct summand (see (2) above for the definition). Indeed, take the idempotent a appearing in the definitions of ζ and μ (see equations (7), (8)) to be the unit element in the field appearing as a direct summand of $QH_{2n}(M)$. Then μ is a Calabi quasi-morphism and ζ is a symplectic quasi-state satisfying the vanishing and the invariance properties. The proof of this statement repeats *verbatim* the one of Theorem 3.1. The results of McDuff [27] yield a remarkable class of symplectic manifolds whose quantum homology, while in general being non-semi-simple, admits a decomposition (2): These are one point blow-ups of *non-uniruled* symplectic manifolds (for instance, of the standard symplectic torus $\mathbb{T}^4$ or of any other symplectic manifold of dimension greater than 2 with $[\omega]|_{\pi_2(M)} = 0$) – see Section 7 for precise definitions and statements.

VARIOUS VERSIONS OF SEMI-SIMPLICITY: Originally the construction of μ and ζ was carried out in [12, 13] for a spherically monotone M under the assumption that the even-degree quantum homology of M is semi-simple as a commutative algebra over a field of Laurent-type series in one variable with complex coefficients (this field can be identified with the Novikov ring in the monotone case). Theorem 5.1 below states that this assumption is equivalent to semi-simplicity of $QH_{2n}(M)$ in the sense of the present paper.

An immediate difficulty with extending the result of [12] to non-monotone manifolds lies in the fact that in the non-monotone case the Novikov ring is no longer a field. In [35] Ostrover bypasses this difficulty by working only with the graded part of degree $2n = \dim M$ of the quantum homology which is already an algebra over a certain field contained in the Novikov ring – in this way he proves the existence of a Calabi quasi-morphism and a non-linear symplectic quasi-state on $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}P^1 \sharp \overline{\mathbb{C}P^1}$ equipped with non-monotone symplectic forms. In addition to the semi-simplicity Ostrover assumes that the minimal Chern number N of M divides n. The definition of quantum homology introduced in Section 2 above is an algebraic extension of the one

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used by Ostrover: our formal variable q, which is responsible for the Chern class, is the root of degree N of the corresponding variable in Ostrover's setting. This minor modification is nevertheless useful: Ostrover's argument (see [35, Section 5]) can be applied *verbatim* to the proof of Theorem 3.1 without the assumption $N \mid n$. Note also that, when we work with $\mathbb{C}$-coefficients, algebraic extensions respect semi-simplicity (see Proposition 2.1(A) above), and hence semi-simplicity in our setting is equivalent to semi-simplicity in Ostrover's setting.

Remark 3.2. There is another notion of semi-simple quantum (co)homology that has been extensively studied by various authors – see e.g. [4], [5], [11], [26], [42] and the references therein. We do not know whether in any of the examples the semi-simplicity in our sense can be formally deduced from the semi-simplicity in the other sense.

$$\boxed{\textbf{From now on we work with the basic field } \mathcal{F} = \mathbb{C}.}$$

APPLICATION TO SYMPLECTIC TORIC FANO 4-MANIFOLDS:

Theorem 3.3. *If M is a symplectic toric Fano 4-manifold with a rational symplectic form, the algebra $QH_4(M)$ is semi-simple.*

QUESTION. Is it true that $QH_{2n}(M)$ is semi-simple for any symplectic toric Fano manifold M^{2n}?

The proof of Theorem 3.3 proceeds as follows. We inspect the list of all symplectic toric Fano 4-manifolds – up to rescaling the symplectic form by a constant factor, there are only five of them: $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{C}P^2$ and the blow-ups of $\mathbb{C}P^2$ at $1, 2, 3$ points. For $\mathbb{C}P^2$ semi-simplicity is elementary (see e.g. [12]), and for $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}P^2$ blown up at one point it was established by Y.Ostrover in [35]. The cases of the blow-ups of $\mathbb{C}P^2$ at 2 or 3 points are new. For the calculations of the quantum homology, we use Batyrev's algorithm (see [3], cf. [30]) which involves the combinatorics/geometry of the moment polytope, see Section 4 for the details.

Theorem 1.1 follows immediately from Theorem 3.1 and Theorem 3.3.

In addition to symplectic toric Fano 4-manifolds with a rational symplectic structure other symplectic manifolds with semi-simple quantum homology include complex projective spaces and complex Grassmannians [12]. We

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show (see Theorem 6.1 below) that by taking products one can construct more examples of manifolds with semi-simple quantum homology. Moreover, by taking direct products of manifolds whose quantum homology contains a field as a direct summand one can construct more manifolds with the same property (see Theorem 6.2 below).

4 Semi-simplicity of $QH_4(M)$ for symplectic toric Fano M^4

4.1 Classification of symplectic toric Fano 4-manifolds

Any closed symplectic toric manifold (M, ω) can be equipped canonically with a complex structure J invariant under the torus action and also compatible with ω (meaning that $\langle \cdot, \cdot \rangle := \omega(\cdot, J\cdot)$ is a J-invariant Riemannian metric). Such a J is unique up to an equivariant biholomorphism – see e.g. [24], Section 9. We say that the symplectic toric manifold (M, ω) is *Fano* if (M, J) is Fano as a complex variety. The symplectic toric Fano 4-manifolds can be completely described as follows:

Proposition 4.1. *If M is a Fano symplectic toric 4-manifold then M is equivariantly symplectomorphic to one of the following toric manifolds: $\mathbb{S}^2 \times \mathbb{S}^2$ or the blow-up of $\mathbb{C}P^2$ at k, $0 \leq k \leq 3$, points, where the symplectic toric structures on $\mathbb{S}^2$, $\mathbb{C}P^2$ are the standard ones and the blow-ups are equivariant (but the sizes of the blow-ups may vary).*

Proof. Denote by J the $\mathbb{T}^2$-invariant complex structure on M compatible with ω. It is known (see e.g. [21, Appendix 1]) that (M, J) is a toric variety whose fan, say Σ, is determined by the moment polytope Δ of the Hamiltonian $\mathbb{T}^2$-action as follows: To i-th edge of Δ assign a perpendicular primitive integral vector e_i pointing inside the polytope. View each e_i as a vector in the dual space $\mathbb{R}^{2*}$. The fan $\Sigma \subset (\mathbb{R}^2)^*$ is defined by the rays ℓ_i generated by e_i's.

Since (M, J) is Fano, the fan Σ is spanned by the edges of *a Fano polytope*, say P. Recall that a convex polytope in $(\mathbb{R}^2)^*$ is called Fano if it contains the origin in its interior, its vertices lie in the integral lattice $\mathbb{Z}^2 \subset (\mathbb{R}^2)^*$ and each pair of consecutive vertices forms a basis of $\mathbb{Z}^2$ (see [16, p.304]). Since the vertices of P are primitive integer vectors lying in ℓ_i's, they coincide with e_i's. We conclude that the vectors e_i form the set of vertices of a Fano

polytope. All such polytopes are classified – there are precisely 5 of them, up to the action of $GL(2, \mathbb{Z})$ (see [23], cf. [16, p.192]), and their fans define the complex toric surfaces listed in the proposition. Looking at polygons Δ having e_i's as inward normals, we get that they correspond to the moment polygons of the standard symplectic toric structures on these surfaces. Thus, by the Delzant theorem [10], (M, ω) is equivariantly symplectomorphic to one of these models. $\qquad\square$

Let us now prove Theorem 3.3. The semi-simplicity of the quantum homology algebra for $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{C}P^2$, $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ was already proved in [12], [35]. Thus it remains to show the claim only for the blow-ups of $\mathbb{C}P^2$ at 2 and 3 points.

4.2   A recipe for computing $QH_*(M)$ for a symplectic toric Fano M^4

Note that a toric manifold has only even-dimensional integral homology and therefore $QH_*(M) = QH_{ev}(M)$. The recipe for computing the quantum homology of a symplectic toric Fano 4-manifold can be extracted from [3], [30]. We shall present it right now in a somewhat simplified form assuming that $M \neq \mathbb{C}P^2$. Consider a moment polytope Δ of M. Let l denote the number of its facets. To each facet Δ_i, $i = 1, \dots, l$, of Δ assign a perpendicular primitive integral vector $e_i = (\alpha_i, \beta_i)$ pointing inside the polytope and a two-dimensional homology class u_i represented by the pre-image of the facet under the moment map. Write the equation for the line containing Δ_i as $\alpha_i x + \beta_i y = \eta_i$, where x, y are the coordinates on the space $\mathbb{R}^2$ where Δ lies. The classes u_i are additive generators of $H_2(M; \mathbb{Z})$ and multiplicative generators of $QH_*(M)$. There are two additive relations between these generators:

$$\sum_{i=1}^{l} \alpha_i u_i = 0, \;\; \sum_{i=1}^{l} \beta_i u_i = 0.$$

The multiplicative relations can be described in the following way. First of all, these relations correspond precisely to primitive subsets $\{i, j\}$ of $\{1, \dots, l\}$ – the definition of a primitive subset in the case $\dim M = 4$, $M \neq \mathbb{C}P^2$, can be given as follows: a subset $I = \{i, j\}$ of $\{1, \dots, l\}$ is called *primitive* if the facets Δ_i and Δ_j do not intersect. Given a primitive set I, set $w := \sum_{i \in I} e_i$. Let $\Sigma \subset (\mathbb{R}^2)^*$ be the fan spanned by e_i's (see the proof of Proposition 4.1

above). Consider the minimal cone in Σ that contains w and assume that the cone is spanned by vectors e_j, $j \in J$ (if $w = 0$ and the cone is zero-dimensional we set $J = \emptyset$). One can show that the set J is always disjoint from I. If $w \neq 0$ then it can be uniquely represented as $w = \sum_{j \in J} c_j e_j$, $c_j \in \mathbb{N}$. Following [3], define an integral vector $d = (d_1, \ldots, d_l)$ as follows: $d_k = 1$ if $k \in I$, $d_k = -c_k$ if $k \in J$ and $d_k = 0$ otherwise[6]. Then the multiplicative relation in $QH_*(M)$ corresponding to I is

$$u_i * u_j = s^{\sum_{k=1}^l d_k \eta_k} q^{-\sum_{k=1}^l d_k} \prod_{m \notin I} u_m^{-d_m}.$$

Finally we note that $QH_4(M)$ is generated, as a subring of $QH_*(M)$, by the elements qu_i.

In what follows we implement the recipe above for equivariant blow-ups of $\mathbb{C}P^2$ at two and three points. We assume that the standard symplectic form on $\mathbb{C}P^2$ is normalized in such a way that the integral over the line equals 1, and consider the standard Hamiltonian torus action on $\mathbb{C}P^2$ whose moment polytope is the triangle Π with the vertices $(0,0), (1,0)$ and $(0,1)$ in $\mathbb{R}^2$. These vertices correspond to the fixed points of the action denoted respectively by P_{00}, P_{10} and P_{01}.

4.3 The case of a blow-up of $\mathbb{C}P^2$ at 2 points

Let M be obtained as a result of the equivariant blow-up of $\mathbb{C}P^2$ at the fixed points P_{10} and P_{01} of the standard action, so that the symplectic areas of the exceptional divisors are equal to $1 - \delta$ and $1 - \epsilon$, where $\epsilon, \delta \in (0; 1)$ are rational numbers satisfying $\epsilon + \delta > 1$. The moment polytope Δ of the blown up torus action on M is obtained from the triangle Π by chopping off the corners (see e.g. [22, Section 3]) at the vertices $(1, 0)$ and $(0, 1)$. Thus Δ is a pentagon with the vertices $(0, 0), (0, \epsilon), (1 - \epsilon, \epsilon), (\delta, 1 - \delta), (\delta, 0)$.

Proposition 4.2. *The $\mathcal{K}$-algebra $QH_4(M)$ is semi-simple.*

Proof. We number the facets of Δ as follows: the facet connecting $(0, \epsilon)$ and $(1 - \epsilon, \epsilon)$ is numbered by 1, then we number the rest of the faces moving

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[6]Note that the definition of d in [30], p.414, does not necessarily lead to a unique d. In order to get the uniqueness one should add that $d_\nu = 0$ unless $\nu \in J$ (with J as defined here). Alternatively, one can drop the uniqueness condition altogether and still get valid multiplicative relations, though a bigger number of them. We thank D.McDuff for clarifying this issue to us.

clockwise from the first one. The vectors e_i are the following: $e_1 = (0, -1)$, $e_2 = (-1, -1)$, $e_3 = (-1, 0)$, $e_4 = (0, 1)$, $e_5 = (1, 0)$. The constants η_i are as follows: $\eta_1 = -\epsilon$, $\eta_2 = -1$, $\eta_3 = -\delta$, $\eta_4 = \eta_5 = 0$.

The additive relations are:

$$-u_2 - u_3 + u_5 = 0, \quad -u_1 - u_2 + u_4 = 0.$$

We choose the basis of $H_2(M; \mathbb{Z})$ as u_1, u_2, u_3, and thus $u_5 = u_2 + u_3$, $u_4 = u_1 + u_2$.

There are five primitive sets: $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 5\}$. Taking into account the additive relations the corresponding multiplicative relations for $QH_*(M)$ can be written as follows:

$$u_1 * u_3 = s^{-(\epsilon+\delta-1)} q^{-1} u_2,$$

$$u_1 * (u_1 + u_2) = s^{-\epsilon} q^{-2} [M],$$

$$u_2 * (u_1 + u_2) = s^{\delta-1} q^{-1} u_3,$$

$$u_2 * (u_2 + u_3) = s^{\epsilon-1} q^{-1} u_1,$$

$$u_3 * (u_2 + u_3) = s^{-\delta} q^{-2} [M].$$

To simplify the notation we shall drop the $*$ sign for the quantum product, and write 1 instead of $[M]$.

We express u_2 from the first relation via u_1, u_3, then define $X := qu_1$, $Y := qu_3$ and substitute the result in the remaining 4 relations. It turns out that X, Y are invertible in $QH_*(M)$ and only 2 of the 4 relations are independent:

$$s^{\epsilon} X^2 (1 + s^{\epsilon+\delta-1} Y) = 1,$$

$$s^{\delta} Y^2 (1 + s^{\epsilon+\delta-1} X) = 1.$$

Factoring $\mathcal{K}[X, Y]$ by these 2 relations yields a complete description of the $\mathcal{K}$-algebra $QH_4(M)$. Expressing Y via X from the first relation and substituting the result in the second shows yields the relation:

$$s^{1-\epsilon} X^5 + (s^{2-2\epsilon-\delta} - 1) X^4 - 2s^{1-2\epsilon} X^3 - 2s^{2-3\epsilon-\delta} X^2 + s^{1-3\epsilon} X + s^{2-4\epsilon-\delta} = 0. \quad (9)$$

Denote the polynomial of X in the left-hand side by $Q_{\epsilon,\delta}(X; s) \in \mathcal{K}[X]$. We have obtained that $QH_4(M) \cong \mathcal{K}[X, X^{-1}]/(Q_{\epsilon,\delta})$. Note that

$$Q_{\epsilon,\delta}(X; 1) = f(X),$$

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where $f(X) := X^5 - 2X^3 - 2X^2 + X + 1$. Denote by $R_{\epsilon,\delta}(s)$ the resultant of $Q_{\epsilon,\delta}(X;s)$ and its derivative with respect to X-variable. Note that $R_{\epsilon,\delta}(1) = r$, where r is the resultant of f and f'. A direct check shows that $r \neq 0$, and hence $R_{\epsilon,\delta}(s) \neq 0 \in \mathcal{K}$. This shows that $Q_{\epsilon,\delta}$ does not have a common root with its derivative for any choice of ϵ and δ, and thus has no multiple roots in any extension of $\mathcal{K}$. Thus the $\mathcal{K}$-algebra $\mathcal{K}[X]/(Q_{\epsilon,\delta})$ is semi-simple. In view of relation (9) X^{-1} is a polynomial of X. Thus $\mathcal{K}[X, X^{-1}]/(Q_{\epsilon,\delta}) = \mathcal{K}[X]/(Q_{\epsilon,\delta})$, and hence $QH_4(M)$ is semisimple. This finishes the proof of the proposition. $\qquad\square$

4.4 The case of a monotone blow-up of $\mathbb{C}P^2$ at 3 points

Let M be obtained as a result of the equivariant blow-up of $\mathbb{C}P^2$ at the fixed points P_{00}, P_{10} and P_{01} of the standard action, so that the symplectic areas of the exceptional divisors are equal to α, $1 - \beta$ and $1 - \gamma$, where $\alpha, \beta, \gamma \in (0; 1)$ are rational numbers satisfying $\alpha < \gamma$, $\alpha < \beta$ and $\beta + \gamma > 1$. The moment polytope Δ of the blown up torus action on M is obtained from the triangle Π by chopping off the corners (see e.g. [22, Section 3]) at its vertices. Thus Δ is a hexagon with the vertices $(\alpha, 0)$, $(0, \alpha)$, $(0, \gamma)$, $(1 - \gamma, \gamma)$, $(\beta, 1 - \beta)$, $(\beta, 0)$.

Proposition 4.3. $QH_4(M)$ is a semi-simple algebra over $\mathcal{K}$.

Proof. By technical reasons, it would be convenient to extend the coefficients of the quantum homology from the field $\mathcal{K} := \mathcal{K}_\Gamma$, where Γ is the group of periods $\Gamma = \mathrm{Span}_{\mathbb{Z}}(1, \alpha, \beta, \gamma)$ (see Section 2), to the field $\bar{\mathcal{K}} := \mathcal{K}_{\mathbb{Q}}$: we put

$$Q := QH_4(M) \otimes_{\mathcal{K}} \bar{K} .$$

In other words, we shall allow any rational powers of the formal variable s responsible for the symplectic area class. In view of Proposition 2.1(A) it suffices to show that Q is semisimple over $\bar{\mathcal{K}}$.

We number the facets of Δ as follows: the facet connecting $(0, \gamma)$ and $(1 - \gamma, \gamma)$ is numbered by 1, then we number the rest of the faces moving clockwise from the first one. The vectors e_i are the following: $e_1 = (0, -1)$, $e_2 = (-1, -1)$, $e_3 = (-1, 0)$, $e_4 = (0, 1)$, $e_5 = (1, 1)$, $e_6 = (1, 0)$. The constants $\eta_1, ..., \eta_6$ are given by

$$-\gamma, -1, -\beta, 0, \alpha, 0 .$$

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We pass to generators of Q by setting $v_i = s^{-\eta_i} q u_i$. There are nine primitive sets: $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{2,6\}$, $\{3,5\}$, $\{3,6\}$, $\{4,6\}$. One readily checks that the multiplicative relations are given by $v_i v_{i+2} = v_{i+1}$ and $v_i v_{i+3} = 1$, for all $i = 1, ..., 6$. Here we set $v_i = v_j$ provided $i = j \mod 6$. Note that
$$v_2 = v_1 v_3, v_4 = v_1^{-1}, v_5 = v_1^{-1} v_3^{-1}, v_6 = v_3^{-1} .$$
Put $A = s^{-1/3} v_1, B = s^{-1/3} v_3$. Put
$$\epsilon = \frac{2}{3} - \gamma, \delta = \frac{2}{3} - \beta, \theta = \alpha - \frac{1}{3} .$$
Put
$$x = s^\epsilon, y = s^\delta, z = s^\theta . \tag{10}$$
Incorporating the additive relations
$$-u_1 - u_2 + u_4 + u_5 = 0, \quad -u_2 - u_3 + u_5 + u_6 = 0$$
we see that $Q = \bar{\mathcal{K}}[A, B]/I$ where I is the ideal given by two polynomial equations
$$A^2 B^2 + x A^2 B - B - z = 0 , \tag{11}$$
and
$$A^2 B^2 + y A B^2 - A - z = 0 . \tag{12}$$
Note that all u_i, and hence A, B are invertible. We wish to show that the ideal I coincides with its radical, which would yield that $\bar{\mathcal{K}}[A, B]/I$ has no nilpotents and is therefore semi-simple. We shall use the following lemma by Seidenberg, see [6], Lemma 8.13: Assume that there exist polynomials $f_A \in I \cap \bar{\mathcal{K}}[A]$ and $f_B \in I \cap \bar{\mathcal{K}}[B]$ so that $gcd(f_A, f'_A) = gcd(f_B, f'_B) = 1$. Then I coincides with its radical. In view of the symmetry between the A- and B-variables, it suffices to produce f_A for all values of ϵ, δ, θ.

Multiply equation (11) by $(A + y)$ and equation (12) by A and take the difference. We get
$$(A + y)(x A^2 - 1)B + (A^2 - yz) = 0 . \tag{13}$$

Multiply this equation by AB and subtract equation (12) multiplied by $(xA^2 - 1)$. We get
$$A(A^2 - yz)B + (A + z)(x A^2 - 1) = 0 . \tag{14}$$

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At this point the proof splits into three cases. We shall use several times the following "non-vanishing test": Let h be a polynomial of n variables with coefficients in $\mathbb{C}$, and let $a_1, ..., a_n$ be real numbers. Then $t = h(s^{a_1}, ..., s^{a_n})$ can be considered as an element of $\bar{\mathcal{K}}$. Specializing t to $s = 1$ we see that $t \neq 0 \in \bar{\mathcal{K}}$ provided $h(1, ..., 1) \neq 0 \in \mathbb{C}$.

Further on, given a polynomial f of the variable A with the coefficients depending on the parameters x, y, z, we denote by f' the derivative with respect to A.

CASE I. Assume that $y \neq z, xyz \neq 1$. Multiply equation (14) by $(A + y)(xA^2 - 1)$ and equation (13) by $A(A^2 - yz)$ and take the difference. We get that

$$f(A; x, y, z) := (A + y)(A + z)(xA^2 - 1)^2 - A(A^2 - yz)^2 = 0 .$$

In other words, $f \in I$ as a polynomial of the variable A for the values of parameters x, y, z given by (10). We claim that $gcd(f, f') = 1$. Denote by $h(x, y, z) \in \mathbb{C}[x, y, z]$ the resultant of f and f'. It suffices to show that

$$h(s^\epsilon, s^\delta, s^\theta) \neq 0 \in \bar{\mathcal{K}} \tag{15}$$

for all ϵ, δ and θ satisfying assumptions of Case I. Using SINGULAR software (see [20]) we calculate that h, interestingly enough, admits a factorization

$$h(x, y, z) = x^2(y - z)^2(xyz - 1)^4 h_0(x, y, z),$$

where h_0 is some explicit polynomial which we do not write here as it contains a large number of monomials. Note that $x \neq 0$, while $y - z$ and $xyz - 1$ do not vanish due to our assumptions. Using SINGULAR again , we calculate that $h_0(1, 1, 1) = 6912 \neq 0$ and hence (15) follows. This completes the proof in Case I.

CASE II. Assume that $y = z, xyz - 1 \neq 0$. Multiply equation (13) by $A(A - y)$ and equation (14) by $(xA^2 - 1)$. Taking into account that $y = z$ and subtracting the obtained equations we get that

$$f(A; x, y) := (A + y)f_0(A; x, y) = 0, \ f_0(A; x, y) = (xA^2 - 1)^2 - A(A - y)^2.$$

Note first that $A = -y$ is not a root of f_0: indeed $f_0(-y; x, y)$ evaluated at $x = s^\epsilon$, $y = s^\delta$ and $s = 1$ equals to $4 \neq 0$. Thus it suffices to show that f_0 is

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mutually prime with f_0'. Denote by $h(x, y) \in \mathbb{C}[x, y]$ the resultant of f_0 and f_0'. It suffices to show that

$$h(s^\epsilon, s^\delta) \neq 0 \in \bar{\mathcal{K}} \tag{16}$$

for all ϵ and δ satisfying assumptions of Case II. Using SINGULAR we calculate that h admits a factorization

$$h(x, y, z) = x^2(xy^2 - 1)^2 h_0(x, y),$$

where $h_0(x, y) = 27x^2y^4 + 256x^3 - 192x^2y - 6xy^2 - 4y^3 + 27$. Note that $x \neq 0$, while $xy^2 - 1$ do not vanish due to our assumptions. Using SINGULAR, we calculate that $h_0(1, 1) = -108 \neq 0$ and hence (16) follows. This completes the proof in Case II.

CASE III. Assume that $xyz = 1$. Multiply equation (13) by $x^{-1}A$ and equation (14) by $(A+y)$. Taking into account that $yz = x^{-1}$ and subtracting the obtained equations we get that

$$f(A; x, y) := x(A^2 - yz)f_0(A; y, z) = 0,$$

$$f_0(A; y, z) = A^2 + (y + z - y^2z^2)A + yz.$$

We claim that f and f' are mutually prime. Substituting $A = \pm\sqrt{yz}$ into f_0 and specializing to $y = z = 1$ we get that $f_0(\pm1; 1, 1) = 2 \pm 1 \neq 0$. Thus $A^2 - yz$ and f_0 have no common roots. The discriminant $d(y, z)$ of f_0 equals

$$d(y, z) = (z + y - y^2z^2)^2 - 4yz .$$

Since $d(1, 1) = -3 \neq 0$ we see that f_0 has no multiple roots. The claim follows. This finishes the proof. $\qquad\square$

This finishes the proof of Theorem 3.3.

5   Quantum homology in the monotone case

We will show now that the semi-simplicity assumption in Theorem 3.1 is equivalent in the monotone case to the one from [12]. Assume M is spherically monotone with $c_1 = \kappa[\omega]$, $\kappa > 0$, on $H_2^S(M)$. In this case one may define the quantum homology ring of M using a different coefficient ring $\widehat{\Lambda}$, which is a

field. The rings $\widehat{\Lambda}$ and Λ are different extensions of some common subring Λ' though none of the rings $\widehat{\Lambda}$, Λ is a subset of another.

The precise definitions are as follows. Denote by N the minimal Chern number of M. Introduce a new variable w, and define Λ' as $\Lambda' = \mathbb{C}[w, w^{-1}]$. Identify Λ' with a subring of Λ through the mapping $w \mapsto (s^\kappa q)^N$. The field $\widehat{\Lambda}$, a different ring extension of Λ', is then defined as

$$\widehat{\Lambda} := \left\{ \sum_{\theta \in \mathbb{Z}} z_\theta w^\theta, \ z_\theta \in \mathbb{C}, \ \sharp\{\theta > c \mid z_\theta \neq 0\} < \infty, \ \forall c \in \mathbb{R} \right\}.$$

Define the grading degree of w as $2N$ and use it to define the grading on $\widehat{\Lambda}$. Note that $\widehat{\Lambda}$ is a field while Λ' is not.

Set $QH'_{ev}(M) := H_{ev}(M; \mathbb{C}) \otimes_{\mathbb{C}} \Lambda'$. The monotonicity of (M, ω) allows to use the Gromov-Witten invariants in order to define the quantum product on $QH'_{ev}(M)$ similarly to the case of $QH_*(M)$ in the following way. Since (M, ω) is spherically monotone $\bar{\pi}_2(M) \cong \mathbb{Z}$. Denote by S the generator of $\bar{\pi}_2(M)$ for which $\omega(S) > 0$. For $a, b \in H_{ev}(M; \mathbb{C})$ and $j \in \mathbb{N}$ we define $(a * b)_j \in H_{ev}(M; \mathbb{C})$ as the unique class satisfying $(a * b)_j \circ c = GW_{jS}(a, b, c)$ for all $c \in H_{ev}(M; \mathbb{C})$, where $\circ$ stands for the ordinary intersection index in homology and $GW_{jS}(a, b, c)$ is the Gromov-Witten invariant which, roughly speaking, counts the pseudo-holomorphic spheres in the homology class jS passing through cycles representing a, b, c. Now for any $a, b \in H_{ev}(M; \mathbb{C})$ set

$$a * b = a \cap b + \sum_{j \in \mathbb{N}} (a * b)_j \ w^{-j}.$$

Since $GW_{jS}(a, b, c) = 0$ unless $\deg a + \deg b + \deg c = 4n - 2Nj$, the sum in the right-hand side is finite and therefore $a * b \in QH'_{ev}(M)$. By $\mathbb{C}$-linearity extend the quantum product to the whole $QH'_{ev}(M)$. As a result, for the same reasons as in the general case, one gets a correctly defined commutative associative product operation on $QH'_{ev}(M)$ [25], [30], [39], [40], [47]. It can be easily seen that $QH'_{ev}(M)$ is a subring of $QH_{ev}(M) = H_{ev}(M; \mathbb{C}) \otimes_{\mathbb{C}} \Lambda$.

Now $\widehat{QH}_{ev}(M) := H_{ev}(M; \mathbb{C}) \otimes_{\mathbb{C}} \widehat{\Lambda}$ and $QH_{ev}(M)$ can be viewed as different extensions (though not subsets of each other) of the ring $QH'_{ev}(M)$:

$$QH_{ev}(M) = QH'_{ev}(M) \otimes_{\Lambda'} \Lambda, \tag{17}$$

$$\widehat{QH}_{ev}(M) = QH'_{ev}(M) \otimes_{\Lambda'} \widehat{\Lambda}, \tag{18}$$

where the tensor product is *the tensor product of rings*. Note that $\widehat{QH}_{ev}(M)$ is a module of finite rank over $\widehat{\Lambda}$.

Similarly to $QH_*(M)$ the ring $\widehat{QH}_{ev}(M)$ gets naturally equipped with a grading. This graded ring is precisely the quantum homology ring used in [12].

Theorem 5.1. *Assume (M, ω) is a closed spherically monotone symplectic manifold. Then $QH_{2n}(M)$ is semi-simple as a $\mathcal{K}$-algebra if and only if $\widehat{QH}_{ev}(M)$ is semi-simple as a $\widehat{\Lambda}$-algebra (which is the semi-simplicity assumption used in [12]).*

Proof. Introduce a new variable u and define a ring

$$\mathcal{R} := \left\{ \sum_{\theta \in \mathbb{Z}} z_\theta u^\theta, \ z_\theta \in \mathbb{C}, \ \sharp\{\theta > c \mid z_\theta \neq 0\} < \infty, \ \forall c \in \mathbb{R} \right\}.$$

The ring $\mathcal{R}$ is graded with the graded degree of u equal 2. Note that $\mathcal{R}$ is a field. We identify $\widehat{\Lambda}$ with a subfield of $\mathcal{R}$ via the mapping $w \mapsto u^N$.

Put $Q := H_{ev}(M; \mathbb{C}) \otimes_\mathbb{C} \mathcal{R}$ – this is a graded $\mathcal{R}$-algebra that can be viewed as a ring extension of $\widehat{QH}_{ev}(M)$:

$$Q = \widehat{QH}_{ev}(M) \otimes_{\widehat{\Lambda}} \mathcal{R},$$

where the tensor product is the tensor product of rings. Denote by Q_{2n} the graded component of degree $2n$ of Q. It is a $\mathbb{C}$-algebra.

Recall that the field $\mathcal{K}$ appearing in the formulation of the theorem is defined as $\mathcal{K}_{\omega(S)\mathbb{Z}}$ (see Section 2 above). Note that $\omega(S) = \kappa N$. Consider an extension $\mathcal{K}_{\kappa\mathbb{Z}}$ of $\mathcal{K} = \mathcal{K}_{\kappa N \mathbb{Z}}$. Introduce two auxiliary rings

$$R_1 = QH_{2n}(M) \otimes_\mathcal{K} \mathcal{K}_{\kappa\mathbb{Z}} \quad \text{and} \quad R_2 = Q_{2n} \otimes_\mathbb{C} \mathcal{K}_{\kappa\mathbb{Z}}.$$

We claim that they are isomorphic as $\mathcal{K}_{\kappa\mathbb{Z}}$-algebras. Indeed, notice that every element of R_1 can be written as $\sum a_j \otimes q^m s^{\kappa j}$ and every element of R_2 as $\sum a_j \otimes u^m s^{\kappa j}$, where in both cases the summation conditions are as follows:

$$j \in \mathbb{Z}, \ a_j \in H_{ev}(M, \mathbb{C}), \ \deg a_j + 2m = 2n,$$

and all a_j vanish for j large enough. The respective multiplications in R_1 and R_2 are given by

$$a *_1 b = \sum_j (a * b)_{jS} q^{-Nj} s^{-\kappa N j}$$

and

$$a *_2 b = \sum_j (a * b)_{jS} u^{-Nj}$$

for all $a, b \in H_{ev}(M, \mathbb{C})$. Define a map $\Psi : R_2 \to R_1$ by

$$\Psi \left(\sum a_j \otimes u^m s^{\kappa j} \right) = \sum a_j \otimes q^m s^{\kappa(m+j)} \; ,$$

where the summation conditions are as above. Roughly speaking, Ψ acts by the substitution $u \mapsto qs^\kappa$. We see that Ψ is a $\mathcal{K}_{\kappa \mathbb{Z}}$-linear map. Furthermore, we have $\Psi(a *_2 b) = a *_1 b$ for all $a, b \in H_{ev}(M, \mathbb{C})$. This readily implies that Ψ is a ring homomorphism. Finally, Ψ is invertible, and the claim follows.

After these preparations let us pass to the proof of the theorem. Assume first that $\widehat{QH}_{ev}(M)$ is semi-simple as a $\widehat{\Lambda}$-algebra. Then, by Proposition 2.1(A), Q is a semi-simple $\mathcal{R}$-algebra. Decompose Q, as a $\mathcal{R}$-algebra, into a direct sum of fields Q_i. The unity e_i of each such field is an idempotent and therefore belongs to Q_{2n}. It is easy to see that Q_{2n} decomposes, as a finite-dimensional $\mathbb{C}$-algebra, into a direct sum of $\mathbb{C}$-algebras $Q_{2n} * e_i$. Each of these algebras is a field as it lies in Q_i. Thus Q_{2n} is a semi-simple $\mathbb{C}$-algebra. Applying Proposition 2.1(A), we get that since Q_{2n} is $\mathbb{C}$-semi-simple, the ring R_2, and hence R_1, is $\mathcal{K}_{\kappa \mathbb{Z}}$-semi-simple. This in turn yields, again by Proposition 2.1(A), the semi-simplicity of $QH_{2n}(M)$ over $\mathcal{K}$.

Assume now that $QH_{2n}(M)$ is semi-simple as a $\mathcal{K}$-algebra. Then by Proposition 2.1(A) R_1, and hence R_2, is $\mathcal{K}_{\kappa \mathbb{Z}}$-semi-simple. Applying again Proposition 2.1(A) we see that Q_{2n} is semi-simple as an $\mathbb{C}$-algebra. Decompose Q_{2n}, as an $\mathbb{C}$-algebra, into a direct sum of fields Q_i with unity elements e_i. Note that all the graded components of Q can be described as $u^i Q_{2n}$, $i \in \mathbb{Z}$. Therefore, as one can easily see, Q decomposes, as an $\mathcal{R}$-algebra, into the direct sum of $\mathcal{R}$-algebras $\mathcal{R} * e_i$ each of which is a field. Thus Q is semi-simple as an $\mathcal{R}$-algebra and hence, by Proposition 2.1(A), $\widehat{QH}_{ev}(M)$ is semi-simple as a $\widehat{\Lambda}$-algebra. This completes the proof. $\square$

6 Products

The following theorem shows how to construct new examples of manifolds with semi-simple quantum homology from the old ones. We continue working within the class $\mathcal{S}$ of symplectic manifolds defined in Section 3 above.

Theorem 6.1. *Assume that closed symplectic manifolds $(M_i^{2n_i}, \omega_i)$, $i = 1, 2$, and $(M, \omega) = (M_1 \times M_2, \omega_1 \oplus \omega_2)$ belong to the class $\mathcal{S}$ and that at least one of the manifolds M_i satisfies the condition $H_k(M_i; \mathbb{C}) = 0$ for all odd k. Assume also that each $QH_{2n_i}(M_i)$, $i = 1, 2$, is semi-simple (as an algebra over $\mathcal{K}_{\Gamma(M_i, \omega_i)}$). Then $QH_{2n_1+2n_2}(M)$ is also semi-simple (as an algebra over $\mathcal{K}_{\Gamma(M, \omega)}$).*

Proof. Let us denote for brevity $\Gamma_i := \Gamma(M_i, \omega_i)$, $i = 1, 2$, and $\Gamma := \Gamma(M, \omega)$. Note that $\Gamma = \Gamma_1 + \Gamma_2$.

Let E_i be a ring which is also a module over Λ_{Γ_i}, $i = 1, 2$. Note that

$$\Lambda_{\Gamma_1 + \Gamma_2} = \Lambda_{\Gamma_i} \otimes_{\mathcal{K}_{\Gamma_i}} \mathcal{K}_{\Gamma_1 + \Gamma_2}, \ i = 1, 2 .$$

We define a modified tensor product operation $\widehat{\otimes}_\Lambda$ by

$$E_1 \widehat{\otimes}_\Lambda E_2 := \left(E_1 \otimes_{\mathcal{K}_{\Gamma_1}} \mathcal{K}_{\Gamma_1 + \Gamma_2} \right) \otimes_{\Lambda_{\Gamma_1 + \Gamma_2}} \left(E_2 \otimes_{\mathcal{K}_{\Gamma_2}} \mathcal{K}_{\Gamma_1 + \Gamma_2} \right).$$

All the tensor products above are automatically assumed to be tensor products of rings. In simple words, we extend both rings to $\Lambda_{\Gamma_1 + \Gamma_2}$-modules and consider the usual tensor product over $\Lambda_{\Gamma_1 + \Gamma_2}$.

Similarly, given algebras F_i, $i = 1, 2$, over $\mathcal{K}_{\Gamma_i}$ one can define their modified tensor product $\widehat{\otimes}_\mathcal{K}$, which is an *algebra* over $\mathcal{K}_\Gamma$, as

$$F_1 \widehat{\otimes}_\mathcal{K} F_2 := \left(F_1 \otimes_{\mathcal{K}_{\Gamma_1}} \mathcal{K}_{\Gamma_1 + \Gamma_2} \right) \otimes_{\mathcal{K}_{\Gamma_1 + \Gamma_2}} \left(F_2 \otimes_{\mathcal{K}_{\Gamma_2}} \mathcal{K}_{\Gamma_1 + \Gamma_2} \right),$$

where again all middle tensor products are tensor products of rings. Proposition 2.1 implies that if each F_i, $i = 1, 2$, is a semi-simple algebra over $\mathcal{K}_{\Gamma_i}$ then $F_1 \widehat{\otimes}_\mathcal{K} F_2$ is a semi-simple algebra over $\mathcal{K}_\Gamma$. As we will show

$$QH_{2n_1+2n_2}(M) = QH_{2n_1}(M_1) \widehat{\otimes}_\mathcal{K} QH_{2n_2}(M_2), \tag{19}$$

which would yield the required semi-simplicity of $QH_{2n_1+2n_2}(M)$ as a $\mathcal{K}_\Gamma$-algebra.

Let us prove (19). Since all the odd-dimensional homology groups of at least one of the manifolds M_1, M_2 are zero, the Künneth formula for quantum homology over the Novikov ring (see e.g. [30, Exercise 11.1.15] for the statement in the monotone case; the general case in our algebraic setup can be treated similarly) yields a *ring isomorphism*

$$QH_{ev}(M) = QH_{ev}(M_1) \widehat{\otimes}_\Lambda QH_{ev}(M_2) .$$

Note that to each element of $QH_{2n_1+2n_2}(M_1 \times M_2)$ of the form $q^m A \otimes B$ (where $A \otimes B$ denotes the classical tensor product of two singular homology classes of *even* degree), one can associate in a unique way a pair of integers k, l with $k + l = m$ so that $q^k A \in QH_{2n_1}(M_1)$ and $q^l B \in QH_{2n_2}(M_2)$. Indeed, the degrees of A and B are even and $\deg q = 2$. Hence we get the required isomorphism (19). □

The proof above can be easily modified using Proposition 2.2 to yield the following result:

Theorem 6.2. *Assume that closed symplectic manifolds $(M_i^{2n_i}, \omega_i)$, $i = 1, 2$, and $(M, \omega) = (M_1 \times M_2, \omega_1 \oplus \omega_2)$ belong to the class $\mathcal{S}$ and that at least one of the manifolds M_i satisfies the condition $H_k(M_i; \mathbb{C}) = 0$ for all odd k. Also assume that each $QH_{2n_i}(M_i)$, $i = 1, 2$, contains a field as a direct summand. Then $QH_{2n_1+2n_2}(M)$ also contains a field as a direct summand.*

7 McDuff's examples

In this section we present McDuff's examples of symplectic manifolds whose quantum homology contains a field as a direct summand but is not semi-simple [28]. Let (M^{2n}, ω) be a closed symplectic manifold. Set the base field $\mathcal{F} = \mathbb{C}$. Set $p := [point] \in H_0(M)$. The manifold M is called *(symplectically) non-uniruled* [27] if the genus zero Gromov-Witten invariant $\langle p, a_2, \ldots, a_k \rangle_{k,\beta}$ of M is zero for any $k \geq 1$ and any $0 \neq \beta \in H_2(M)$, $a_i \in H_*(M)$. Here the marked points in the definition of the Gromov-Witten invariant are allowed to vary freely. In particular, the genus zero 3-point Gromov-Witten invariants $GW_\beta^{\mathbb{C}}(p, a_2, a_3)$ involved in the definition of the quantum homology of M over $\mathbb{C}$ (see Section 2) are zero for all $0 \neq \beta \in H_2(M)$, $a_2, a_3 \in H_*(M)$.

The class of non-uniruled symplectic manifolds includes the manifolds satisfying $[\omega]|_{\pi_2(M)} = 0$ or Kähler manifolds with $c_1 < 0$ (for instance, complex hypersurfaces in $\mathbb{C}P^n$ of degree greater than $n + 1$).

Let $\widetilde{M}$ be a one point blow-up of a non-uniruled symplectic manifold (M^{2n}, ω), $n \geq 2$. Denote by $\tilde{\omega}$ the symplectic form on $\widetilde{M}$, by E the homology class of the exceptional divisor in $\widetilde{M}$. We will write E^i, $i = 1, ..., n - 1$ for the i-th power of E with respect to the classical cap-product. Introduce the parameter of the blow-up, $\delta := [\tilde{\omega}](E^{n-1})$.

We wish M and $\widetilde{M}$ to stay within the class $\mathcal{S}$ of symplectic manifolds introduced in Section 3 above. This holds, for instance, either if $\dim M = 4$, $[\omega]$ is rational and a non-zero rational multiple of δ lies in the group of periods $\Gamma(M, \omega)$, or if $\pi_2(M) = 0$.

Our goal is to deduce from McDuff's computation of $QH_*(\widetilde{M})$ [27] the following result (an equivalent formulation was suggested to us by McDuff [28]):

Theorem 7.1. *If M is non-uniruled, the algebra $QH_{2n}(\widetilde{M})$ contains a field as a direct summand, but is not semi-simple.*

Convention: In what follows, we use notation $\dotplus$ for the direct sum in the category of modules, and $\oplus$ for the direct sum in the category of algebras.

Proposition 7.2 ([27], Cor. 2.4). *Let $0 \le i, j < n$. Then*

$$E^i * E^j = E^i \cap E^j = E^{i+j}, \quad if \quad i + j < n,$$

$$E^i * E^{n-i} = -p + E \otimes q^{-n+1} s^{-\delta} .$$

Denote $\widetilde{\mathcal{K}} = \mathcal{K}_{\Gamma(\widetilde{M}, \tilde{\omega})}$. Set $\mathbf{1} := [\widetilde{M}]$, $A := Eq$, $B := pq^n$. Let us define the following $\widetilde{\mathcal{K}}$-submodules of $QH_{2n}(\widetilde{M})$:

$$U := \operatorname{span}_{\widetilde{\mathcal{K}}}(A, A^2, \dots, A^{n-1}),$$

where the powers A^i are taken with respect to the quantum product,

$$W := QH_{2n}(\widetilde{M}) \cap \left(\dotplus_{0 < i < 2n} H_i(M) \otimes \widetilde{\Lambda} \right).$$

Thus additively

$$QH_{2n}(\widetilde{M}) = U \dotplus \widetilde{\mathcal{K}} B \dotplus \widetilde{\mathcal{K}} \mathbf{1} \dotplus W.$$

As for the multiplicative structure of $QH_{2n}(\widetilde{M})$, D.McDuff proved the following

Proposition 7.3 ([27], Prop. 2.5, Cor. 2.6, Lem. 3.3).

$$U * W = 0,$$

$$W * W \subset W \dotplus \widetilde{\mathcal{K}} B,$$

$$B * (U \dotplus W) = 0,$$

$$B * B = 0.$$

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PROOF OF THEOREM 7.1. Set $V := U \dotplus \widetilde{\mathcal{K}}B \dotplus \widetilde{\mathcal{K}}\mathbf{1}$. Then one has $QH_{2n}(\widetilde{M}) = V \dotplus W$. By Propositions 7.2 and 7.3, V is closed under the quantum product. It is multiplicatively generated over $\widetilde{\mathcal{K}}$ by A, B and $\mathbf{1}$ with the relations:

$$A^n = -B + As^{-\delta},$$

$$A * B = 0,$$

see Propositions 7.2, 7.3. Expressing B via A from the first relation and substituting the result into the second one we obtain the following isomorphism of $\widetilde{\mathcal{K}}$-algebras:

$$V \cong \widetilde{\mathcal{K}}[A]/(f),$$

where (f) is an ideal in $\widetilde{\mathcal{K}}[A]$ generated by $f(A) := A^2(A^{n-1} - s^{-\delta})$.

Set $g(A) := A^2$, $h(A) := A^{n-1} - s^{-\delta}$, so that $f = gh$. Decompose h into mutually prime polynomials $P_1, \ldots, P_k \in \widetilde{\mathcal{K}}[A]$, where each P_i is irreducible over $\widetilde{\mathcal{K}}$ (we can choose P_i's to be irreducible – and not powers of irreducible polynomials – since h has no common factors with its derivative). By the Chinese remainder theorem, there exists an isomorphism of $\widetilde{\mathcal{K}}$-algebras

$$I : V = \widetilde{\mathcal{K}}[A]/(f) \to \widetilde{\mathcal{K}}[A]/(g) \oplus \widetilde{\mathcal{K}}[A]/(P_1) \oplus \ldots \oplus \widetilde{\mathcal{K}}[A]/(P_k),$$

where I sends each $u(A) + (f) \in \widetilde{\mathcal{K}}[A]/(f)$ to the direct sum of the remainders of the polynomial $u(A)$ modulo $g, P_1, \ldots, P_k$. Since each P_i is irreducible over $\widetilde{\mathcal{K}}$, the factor algebra $\widetilde{\mathcal{K}}[A]/(P_i)$ is a field. Let $F \subset V$ be a field which is the preimage of $\widetilde{\mathcal{K}}[A]/(P_k)$ under the isomorphism I above. The field F consists of all those remainders $u \in \widetilde{\mathcal{K}}[A]$ mod f that are divisible by $g, P_1, \ldots, P_{k-1}$. In particular, each such u is divisible by $g(A) = A^2$. In view of Proposition 7.3 this yields

$$F * W = 0, \ F * B = 0 . \tag{20}$$

Put

$$Z = I^{-1}\left(\widetilde{\mathcal{K}}[A]/(g) \oplus \widetilde{\mathcal{K}}[A]/(P_1) \oplus \ldots \oplus \widetilde{\mathcal{K}}[A]/(P_{k-1})\right) .$$

Since $V = F \oplus Z$ we have

$$F * Z = 0, \ Z * Z \subset Z . \tag{21}$$

Look at the decomposition

$$QH_{2n}(\widetilde{M}) = F \dotplus Z \dotplus W .$$

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Let us show that in fact

$$QH_{2n}(\widetilde{M}) = F \oplus (Z \dot{+} W) \,. \tag{22}$$

Indeed, by formulas (21) and (20) we have $F * (Z \dot{+} W) = 0$. Thus it suffices to show that $Z \dot{+} W$ is closed under the quantum multiplication.

Note that $Z * Z \subset Z$ by (21). Further, since $F * B = 0$ by (20), the element $B \in V$ necessarily lies in Z. Together with Proposition 7.3 this implies

$$W * W \subset \widetilde{\mathcal{K}} B \dot{+} W \subset Z \dot{+} W. \tag{23}$$

Finally,

$$Z \subset V = U \dot{+} \widetilde{\mathcal{K}} B \dot{+} \widetilde{\mathcal{K}} \mathbf{1}$$

and hence by Proposition 7.3

$$Z * W \subset W \,.$$

This yields decomposition (22) which tells us that $QH_{2n}(\widetilde{M})$ contains a field F as a direct summand.

At the same time $QH_{2n}(\widetilde{M})$ is not semi-simple since it contains a nilpotent element B. This completes the proof. $\qquad\square$

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